11/25 Lecture Notes

Theorem: Let A be an $n \times n$ matrix w/ eigenvalue λ. The set of all eigenvectors w/ eigenvalue λ along w/ $\vec{0}$ is a subspace of $\mathbb{R}^n$.

Proof: This is $\text{null}(A - \lambda I)$

eigenvectors are nonzero

Definition: This is called the eigenspace of λ

Theorem: λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$

Definition: $\det(A - \lambda I)$ is called characteristic polynomial of A

Ex What are the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 0 & 9 & 3 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 0 & 9 & 3 \\ 0 & 0 & 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}\right) = 0 \to \det\left(\begin{bmatrix} 4-\lambda & 1 & 2 & 1 \\ 0 & 3-\lambda & 4 & 2 \\ 0 & 0 & 9-\lambda & 3 \\ 0 & 0 & 2 & 4-\lambda \end{bmatrix}\right) = 0$$

$\Rightarrow (4-\lambda)(3-\lambda)((9-\lambda)(4-\lambda) - 6) = 0$

$\Rightarrow (4-\lambda)(3-\lambda)^2(10-\lambda) = 0$

So, $\boxed{\underbrace{\lambda = 4, 3, 10}_{\text{eigenvalues}}}$ $\Rightarrow$ 3 here has **multiplicity 2** bc it shows up twice

For 3: $\begin{bmatrix} 1 & 1 & 2 & 1 & | & 0 \\ 0 & 0 & 4 & 2 & | & 0 \\ 0 & 0 & 6 & 3 & | & 0 \\ 0 & 0 & 2 & 1 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 2 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$ = corresponding eigenspace $\Rightarrow$ (forms a basis)

For 4: $\begin{bmatrix} 0 & 1 & 2 & 1 & | & 0 \\ 0 & -1 & 4 & 2 & | & 0 \\ 0 & 0 & 5 & 3 & | & 0 \\ 0 & 0 & 2 & 0 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow x_{2 \to 4} = 0,\ x_1 = s_1 \Rightarrow$ so: $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$

For 10: $\begin{bmatrix} -6 & 1 & 2 & 1 & | & 0 \\ 0 & -7 & 4 & 2 & | & 0 \\ 0 & 0 & -1 & 3 & | & 0 \\ 0 & 0 & 2 & -6 & | & 0 \end{bmatrix} \sim \begin{bmatrix} -6 & 1 & 2 & 1 & | & 0 \\ 0 & -7 & 4 & 2 & | & 0 \\ 0 & 0 & -1 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \Big\} \vec{x} = \begin{bmatrix} 3/2 \\ 2 \\ 3 \\ 1 \end{bmatrix} s_1 \Rightarrow \text{span}\left(\begin{bmatrix} 3 \\ 4 \\ 6 \\ 2 \end{bmatrix}\right)$

Definition: A root α of a polynomial equation $p(x) = 0$ has multiplicity r if $p(x) = (x-\alpha)^r q(x)$ and $q(\alpha) \neq 0$

Ex $p(x) = (x-1)^2(x-2)^3(x-3)$

multiplicity of $1 = 2$, multiplicity of $2 = 3$, multiplicity of $3 = 1$,

multiplicity of $4 = 0$

Ex What are the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & 2 \\ -3 & 1 & 5 \end{bmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow \det\begin{bmatrix} 1-\lambda & 1 & 1 \\ -2 & 2-\lambda & 2 \\ -3 & 1 & 5-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)((-1)^2 \det\begin{bmatrix} 2-\lambda & 2 \\ 1 & 5-\lambda \end{bmatrix})$

$+ ((-1)^3 \det\begin{bmatrix} -2 & 2 \\ -3 & 5-\lambda \end{bmatrix}) + ((-1)^4 \det\begin{bmatrix} -2 & 2-\lambda \\ -3 & 1 \end{bmatrix}) = 0$

$(1-\lambda)((2-\lambda)(5-\lambda) - 2) - (-2(5-\lambda) + 6) + (-2 + 3(2-\lambda)) = 0$

$(1-\lambda)(2-\lambda)(5-\lambda) - 2(1-\lambda) + 2(5-\lambda) - 6 - 2 + 3(2-\lambda) = 0$

$(2-\lambda)((1-\lambda)(5-\lambda) + 3) = 0$

$(2-\lambda)(\lambda^2 - 6\lambda + 8) = 0$ $\to (2-\lambda)(2-\lambda)(4-\lambda) = 0 \to$ (next page)

11/25 Lecture Notes (continued)

$\lambda = 2, 4 \Rightarrow$ (2 has mult 2, 4 mult 1)

$\lambda = 2$: $\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ -2 & 0 & 2 & 0 \\ -3 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

multiplicity of eigenvalue $\neq$ dimension of eigenspace

multiplicity = upper bound of "

$\vec{x} = \begin{bmatrix}1\\0\\1\end{bmatrix} s_1 \Rightarrow$ eigenspace for $\lambda = 2 \Rightarrow$ span$\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$

Theorem: Let A be an $n \times n$ matrix w/ eigenvalue λ. Then the dimension of the eigenspace associated w/ λ is $\leq$ multiplicity of λ.

Final Version of the Big Unifying Theorem:

Let $\mathcal{A} = \{\vec{a}_1, \ldots, \vec{a}_n\}$ be vectors in $\mathbb{R}^n$, $A = [a_1, \ldots, a_n]$, $T: \mathbb{R}^n \to \mathbb{R}^n$ be given $T(\vec{x}) = A\vec{x}$. Then the following are equivalent (all are true if 1 is true, & all are false if 1 is false):

a) $\mathcal{A}$ spans $\mathbb{R}^n$

b) $\mathcal{A}$ is linearly independent

c) $A\vec{x} = \vec{b}$ has a unique solution for all b in $\mathbb{R}^n$

d) T is onto

e) T is 1-1

f) A is invertible

g) $\ker(T) = 0$

h) $\mathcal{A}$ is a basis for $\mathbb{R}^n$

i) $\mathrm{col}(A) = \mathbb{R}^n$

j) $\mathrm{row}(A) = \mathbb{R}^n$

k) $\mathrm{rank}(A) = n$

l) $\det(A) \neq 0$

m) 0 is not an eigenvalue.